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The effect of variable properties on momentum and heat transfer in a tube with constant wall temperature†

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Abstract—The effect of variable properties on momentum and heat transfer is investigated applying an asymptotic method that had been used before with the thermal boundary condition q_w = const. Though the changes are in the thermal boundary condition only it turns out that both cases are quite different in several aspects. As a main result it turns out that a method empirically used in both cases (property ratio method) is not uniquely valid in the T_w = const. case of this study.

1. INTRODUCTION

IN AN EARLIER study [1], the influence of variable properties on momentum and heat transfer has been determined for the thermal boundary condition of *constant heat flux across the wall.* In this study a regular perturbation technique was applied with the final results given in the form of the so-called property ratio method. This original empirical method accounts for the influence of variable properties by certain property ratios in the skin friction and Nusselt number results. For example the skin friction coefficient f is

$$f = f_{cp}\left\{\left[\frac{\rho_w^*}{\rho_B^*}\right]^{n_\rho}\left[\frac{\eta_w^*}{\eta_B^*}\right]^{n_\eta}\right\}$$

where f_{cp} is the friction factor for constant properties and subscripts w, B the wall and bulk conditions, respectively. Applying the perturbation technique the exponents n_ρ and n_η were determined analytically from the basic equations. For constant wall heat flux the analytical results compare well with empirically determined exponents [1].

The objective of the present study is an extension of the previous work now prescribing a constant wall temperature downstream of $x^* = 0$, which is different from the wall temperature of the oncoming isothermal flow.

The authors expected results that differ quantitatively rather than qualitatively compared with those for q_w^* = const. These expectations were backed by empirical findings. So, for example Kays and Crawford (p. 279 of ref. [2]) summarize with respect to the exponents in the property ratio method: "Also there appears to be little effect due to different types of thermal boundary conditions".

The subsequent study will show that this is not true.

† Dedicated to Prof. Dr.-Ing. Klaus Gersten on the occasion of his 60th birthday.

2. FLOW SITUATION; BASIC EQUATIONS

As in the q_w^* = const. case laminar tube flow will be considered, which is *fully developed* in the constant property limit. There are two aspects of the 'fully developed' condition: hydrodynamically it means that the velocity profile remains unchanged throughout the whole pipe flow. Thermally fully developed refers to a flow in which the temperature profile in the T_w^* = const. case meets a condition illustrated in Fig. 1. Downstream of $x^* = 0$, after a certain thermal adjustment zone all temperature profiles exhibit a similarity in the defect profile $T_w^* - T^*$, which decays exponentially with x^*. It turns out that this condition holds asymptotically for $x^* \to \infty$ only, but for practical applications it is a good approximation for finite x^*. Further details will be given in Section 5 (equation (46)).

For the flow under consideration the basic equations are the Navier–Stokes equations for slender channels [3].

Neglecting viscous heating ($Ma \to 0$) and buoyancy forces ($Fr \to 0$) they read, nondimensionalized and transformed according to Table 1

$$\frac{\partial}{\partial x}(\rho u) + \frac{1}{r}\frac{\partial}{\partial r}(\rho r v) = 0 \tag{1}$$

$$\rho\left(u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r}\right) = -\frac{\partial p}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\eta\frac{\partial u}{\partial r}\right) \tag{2}$$

$$0 = -\frac{\partial p}{\partial r} \tag{3}$$

$$\rho\left(u\frac{\partial h}{\partial x} + v\frac{\partial h}{\partial r}\right) = \frac{1}{Pr_w}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\lambda}{c_p}\frac{\partial h}{\partial r}\right) \tag{4}$$

with associated boundary conditions

$$u = v = h = 0 \quad \text{at} \quad r = 1 \tag{5}$$

$$v = \frac{\partial u}{\partial r} = \frac{\partial h}{\partial r} = 0 \quad \text{at} \quad r = 0. \tag{6}$$

NOMENCLATURE

A	cross-sectional area
A_α	auxiliary functions, equation (71)
B_α	auxiliary functions, equation (72)
c_p	specific heat at constant pressure
f	friction factor, equation (51)
h	specific enthalpy
K_α	dimensionless property, equation (10)
m_α, n_α	exponents, property ratio method
Nu	Nusselt number, equation (52)
p	pressure
Pr	Prandtl number, equation (7)
q_w	wall heat flux
r	radial coordinate
R	pipe radius
Re	Reynolds number, equation (7)
T	temperature
$\bar{T}$	reduced temperature, equation (13)
u, v	velocity components
U_w	mean velocity at reference conditions
x	axial coordinate.

Greek symbols

α	physical property
ε	perturbation parameter, equation (12)
η	viscosity
λ	thermal conductivity
Λ_1	eigenvalue, equation (46)
ρ	density
τ_w	wall shear stress.

Subscripts

B	bulk
cp	constant property
w	wall
0, 1	zero, first order
∞	upstream conditions
α	associated with the property α.

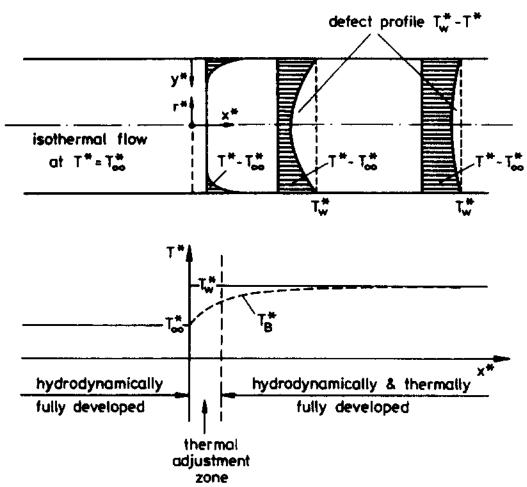

FIG. 1. Development of the temperature profile, constant properties.

All variables are nondimensionalized with quantities at the reference state 'w' far downstream ($x^* \to \infty$).

The Reynolds and Prandtl numbers are define with properties at the reference state

$$Re_w = \frac{\rho_w^* U_w^* R^*}{\eta_w^*}, \quad Pr_w = \frac{\eta_w^* c_{pw}^*}{\lambda_w^*}, \quad U_w^* = \frac{\dot{m}^*}{\rho_w^* A^*}. \tag{7}$$

The Prandtl number is left as the only parameter of equations (1)–(6) since the Reynolds number could be eliminated as an explicit parameter by the transformation according to Table 1.

3. PHYSICAL PROPERTIES

The physical properties involved in the problem are viscosity η, density ρ, thermal conductivity λ, and specific heat capacity c_p. As c_p is not constant the energy equation is written in enthalpy h rather than temperature T. The relation between h and T is

$$\mathrm{d}h = c_p \,\mathrm{d}T \quad \text{or} \quad h = \int_0^T c_p \,\mathrm{d}T. \tag{8}$$

All these properties are (more or less) pressure and temperature dependent so that they can be expanded as a Taylor series at the reference state 'w'. A general physical property α (here: η, ρ, λ or c_p) then reads

$$\alpha^* = \alpha_w^* + \left.\frac{\partial \alpha^*}{\partial T^*}\right|_w (T^* - T_w^*) + \left.\frac{\partial \alpha^*}{\partial p^*}\right|_w (p^* - p_w^*) + \dots \tag{9}$$

It turns out that only the temperature dependence is of practical importance if we impose the restriction of small Mach number, i.e. $Ma \to 0$. (For a detailed discussion see ref. [1].)

Table 1. Dimensionless, transformed variables (*, dimensionless quantities)

x	r	u	v	p	T	h	ρ	η	λ	c_p
$x^*/R^* Re_w$	r^*/R^*	u^*/U_w^*	$v^* Re_w/U_w^*$	$p^*-p_\infty^*/\rho_w^* U_w^{*2}$	$T^*-T_w^*/T_w^*$	$h^*-h_w^*/c_{pw}^* T_w^*$	ρ^*/ρ_w^*	η^*/η_w^*	λ^*/λ_w^*	c_p^*/c_{pw}^*

Table 2. Variation of physical properties with temperature ($T_w^* = 293$ K)

	K_ρ	K_η	K_λ	K_c
Air	−1.0	0.70	0.83	0.01
Water	−0.06	−7.37	0.75	−0.05

With the dimensionless fluid parameter

$$K_\alpha \equiv \left(\frac{T^*}{\alpha^*}\frac{\partial \alpha^*}{\partial T^*}\right)_w \quad (10)$$

the Taylor series (9) then reads (pressure dependence neglected)

$$\alpha \equiv \frac{\alpha^*}{\alpha_w^*} = 1 + K_\alpha T + \ldots \quad (11)$$

In Table 2 values of K_α are listed for air and water. With temperatures close to the reference temperature T_w^* the linear approximation of α will be sufficient. The subsequent theory based on equation (11) is straightforward, see for example ref. [4].

4. ASYMPTOTIC APPROACH

The perturbation parameter is defined as the starting point of our asymptotic approach. The basic flow, which will be perturbed asymptotically, is that for constant properties. Obviously this flow situation prevails for vanishing heat transfer rates, $(T_\infty^* - T_w^*) \to 0$.

Therefore, our perturbation parameter ε is

$$\varepsilon = \frac{T_\infty^* - T_w^*}{T_w^*}. \quad (12)$$

In what follows ε is assumed to be small with $\varepsilon = 0$ for the constant property limit. Deviations from the constant property case are therefore due to non-zero values of ε. By means of the regular perturbation technique this can be formulated systematically.

As a first step equation (11) is rewritten as

$$\alpha = 1 + \varepsilon K_\alpha \bar{T} + O(\varepsilon^2), \quad \bar{T} = \frac{T}{\varepsilon} = \frac{T^* - T_w}{T_\infty^* - T_w^*}. \quad (13)$$

According to this expansion all dependent variables of the problem are assumed to be of a similar form

$$u = u_0 + \varepsilon(K_\eta u_{1\eta} + K_\rho u_{1\rho}) + O(\varepsilon^2) \quad (14)$$

$$v = v_0 + \varepsilon(K_\eta v_{1\eta} + K_\rho v_{1\rho}) + O(\varepsilon^2) \quad (15)$$

$$p = p_0 + \varepsilon(K_\eta p_{1\eta} + K_\rho p_{1\rho}) + O(\varepsilon^2) \quad (16)$$

$$\bar{T} = \bar{T}_0 + \varepsilon(K_\eta \bar{T}_{1\eta} + K_\rho \bar{T}_{1\rho} + K_\lambda \bar{T}_{1\lambda} + K_c \bar{T}_{1c}) + O(\varepsilon^2). \quad (17)$$

All variables with the index 0, like $u_0, v_0, \ldots$, describe the constant property solution, those with index 1, like $u_{1\eta}, v_{1\eta}, \ldots$, are coefficients of the linear (in ε) deviations according to the temperature dependence of the physical properties.

The corresponding sets of equations are derived from the basic equations (1)–(4). After inserting expansions (13)–(17) terms of equal magnitude $(\varepsilon K_\alpha)^n$ with $n = 0$, 1 and $\alpha = \rho, \eta, \lambda, c_p$ are collected.

Zero-order system ($n = 0$, constant properties)

$$\frac{\partial u_0}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r v_0) = 0 \quad (18)$$

$$u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial r} = -\frac{\partial p_0}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_0}{\partial r}\right) \quad (19)$$

$$0 = -\frac{\partial p_0}{\partial r} \quad (20)$$

$$u_0 \frac{\partial \bar{T}_0}{\partial x} + v_0 \frac{\partial \bar{T}_0}{\partial r} = \frac{1}{Pr_w}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \bar{T}_0}{\partial r}\right). \quad (21)$$

First-order system ($n = 1$, linear deviations)

$$\frac{\partial u_{1\eta}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r v_{1\eta}) = 0 \quad (22)$$

$$u_{1\eta} \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_{1\eta}}{\partial x} + v_{1\eta} \frac{\partial u_0}{\partial r} + v_0 \frac{\partial u_{1\eta}}{\partial r} = -\frac{\partial p_{1\eta}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial u_{1\eta}}{\partial r} + \bar{T}_0 \frac{\partial u_0}{\partial r}\right)\right) \quad (23)$$

$$0 = -\frac{\partial p_{1\eta}}{\partial r} \quad (24)$$

$$u_{1\eta} \frac{\partial \bar{T}_0}{\partial x} + u_0 \frac{\partial \bar{T}_{1\eta}}{\partial x} + v_{1\eta} \frac{\partial \bar{T}_0}{\partial r} + v_0 \frac{\partial \bar{T}_{1\eta}}{\partial r} = \frac{1}{Pr_w}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \bar{T}_{1\eta}}{\partial r}\right) \quad (25)$$

$$\frac{\partial u_{1\rho}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(r v_{1\rho}) = -\left(\frac{\partial}{\partial x}(\bar{T}_0 u_0) + \frac{1}{r}\frac{\partial}{\partial r}(r \bar{T}_0 v_0)\right) \quad (26)$$

$$u_{1\rho} \frac{\partial u_0}{\partial x} + u_0 \frac{\partial u_{1\rho}}{\partial x} + v_{1\rho} \frac{\partial u_0}{\partial r} + v_0 \frac{\partial u_{1\rho}}{\partial r} = -\frac{\partial p_{1\rho}}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial u_{1\rho}}{\partial r}\right) - \bar{T}_0\left(u_0 \frac{\partial u_0}{\partial x} + v_0 \frac{\partial u_0}{\partial r}\right) \quad (27)$$

$$0 = -\frac{\partial p_{1\rho}}{\partial r} \quad (28)$$

$$u_{1\rho} \frac{\partial \bar{T}_0}{\partial x} + u_0 \frac{\partial \bar{T}_{1\rho}}{\partial x} + v_{1\rho} \frac{\partial \bar{T}_0}{\partial r} + v_0 \frac{\partial \bar{T}_{1\rho}}{\partial r} = \frac{1}{Pr_w}\frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial \bar{T}_{1\rho}}{\partial r}\right) - \bar{T}_0\left(u_0 \frac{\partial \bar{T}_0}{\partial x} + v_0 \frac{\partial \bar{T}_0}{\partial r}\right) \quad (29)$$

$$u_0 \frac{\partial \bar{T}_{1\lambda}}{\partial x} + v_0 \frac{\partial \bar{T}_{1\lambda}}{\partial r} = \frac{1}{Pr_w}\frac{1}{r}\frac{\partial}{\partial r}\left(r\left(\frac{\partial \bar{T}_{1\lambda}}{\partial r} + \bar{T}_0 \frac{\partial \bar{T}_0}{\partial r}\right)\right) \quad (30)$$

$$u_0\left(\frac{\partial \bar{T}_{1c}}{\partial x}+\bar{T}_0\frac{\partial \bar{T}_0}{\partial x}\right)+v_0\left(\frac{\partial \bar{T}_{1c}}{\partial r}+\bar{T}_0\frac{\partial \bar{T}_0}{\partial r}\right)$$
$$=\frac{1}{Pr_w}\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial \bar{T}_{1c}}{\partial r}\right). \quad (31)$$

Energy equations (21), (25) and (29)–(31) are rewritten in temperatures $\bar{T}_i$. Evaluating equation (8) with $\bar{h} = h/\varepsilon$ for this purpose gives

$$\bar{h}_i = \bar{T}_i \quad \text{with} \quad i = 0, 1\eta, 1\rho, 1\lambda \quad (32)$$

$$\bar{h}_{1c} = \bar{T}_{1c}+\tfrac{1}{2}\bar{T}_0^2. \quad (33)$$

The associated boundary conditions at $r = 0$, 1 and $x \geqslant 0$ from equations (5) and (6) are

$$u_i = v_i = \bar{T}_j = 0 \quad \text{at} \quad r = 1, x \geqslant 0 \quad (34)$$

$$v_i = \frac{\partial u_i}{\partial r} = \frac{\partial \bar{T}_j}{\partial r} = 0 \quad \text{at} \quad r = 0, x \geqslant 0 \quad (35)$$

$$i = 0, 1\eta, 1\rho \quad j = 0, 1\eta, 1\rho, 1\lambda, 1c.$$

The initial conditions at $x = 0$ are

$$u_0 = 2(1-r^2) \quad (36)$$

$$u_{1\eta} = 0 \quad (37)$$

$$u_{1\rho} = -u_0 \quad (38)$$

$$v_i = 0; \quad i = 0, 1\eta, 1\rho \quad (39)$$

$$\frac{\partial p_0}{\partial x} = -8 \quad (40)$$

$$\frac{\partial p_i}{\partial x} = 0; \quad i = 1\eta, 1\rho \quad (41)$$

$$\bar{T}_0 = 1 \quad (42)$$

$$\bar{T}_j = 0; \quad j = 1\eta, 1\rho, 1\lambda, 1c. \quad (43)$$

The zero-order initial conditions (36), (39) and (40) hold since the constant property flow is hydrodynamically fully developed for all locations x. The zero-order temperature condition (42) holds since the upstream flow ($x \leqslant 0$) is an isothermal flow at $T^* = T^*_\infty$ uninfluenced by the temperature jump at $x = 0$.

For the same reason all first-order initial conditions are zero with the exception of equation (38), which is nonzero as a consequence of our choice of reference state. With $\bar{T}_0 = 1$ the density is $\rho = 1+\varepsilon K_\rho+O(\varepsilon^2)$ at $x = 0$. Inserting this into the integral condition of constant mass flux

$$2\int_0^1 \rho u r \,\mathrm{d}r = 1 \quad (44)$$

immediately provides initial condition (38).

5. SOLUTIONS

There are five sets of linear (with the exception of equation (19)) partial differential equations and their corresponding boundary and initial conditions.

One for constant property flow (zero order): equations (18)–(21) together with equations (34)–(36), (39), (40) and (42).

Four for linear deviations with respect to ε (first order), i.e. one set for each of the four physical properties η, ρ, λ and c_p. All equations, boundary and initial conditions can easily be collected from equations (22) to (43) according to the appropriate subscripts.

The common solution procedure for the constant property energy equation is that of separation of variables which reduces the governing differential equation (21) to the Sturm–Liouville type. The solution is then obtained in the form of an infinite series expansion in terms of eigenvalues and eigenfunctions, see for example p. 100 of Shah and London [5]

$$\bar{T}_0 = \sum_{n=1}^{\infty} C_{0n}\bar{T}_{0n}(r)\exp\left[-\Lambda_n^2 x/Pr_w\right] \quad (45)$$

where C_{0n} are constants and Λ_n the corresponding eigenvalues with respect to the eigenfunctions $\bar{T}_{0n}$. For $x \to \infty$ the first term of the infinite series is dominating so that the downstream limit of equation (45) reads

$$\bar{T}_0 = C_{01}\bar{T}_{01}(r)\exp\left[-\Lambda_1^2 x/Pr_w\right]; \quad C_{01} = 1.4764, \; \Lambda_1^2 = 3.6568. \quad (46)$$

Our first approach to the first-order equations was based on the expectation that the exponential x-dependence in equation (46) would be carried to the higher order equations. We therefore assumed higher order solutions of the general form 'function of r' times $\exp[-\Lambda_1^2 x/Pr_w]$, for example

$$u_{1\eta} = \bar{u}_{1\eta}(r)\exp\left[-\Lambda_1^2 x/Pr_w\right]. \quad (47)$$

Actually it is a necessary condition for constant exponents in the property ratio method that the first-order velocity functions $u_{1\eta}, u_{1\rho}, \ldots$ are of this general type (see equation (69) below).

It turned out that this only holds for large Prandtl numbers (asymptotically for $Pr_w \to \infty$, see Section 5.2 below). We therefore solved all equations in a completely numerical approach. Based on these numerical solutions we return to our first attempt and try to interpret its failure by physical arguments in Section 5.2.

One consequence should be mentioned here: constant exponents in the property ratio method (in contrast to the q_w = const. case) do not exist for arbitrary Prandtl numbers!

5.1. *Completely numerical approach*

The zero-order (constant property) flow solution is well known in its analytical form

$$u_0 = 2(1-r^2); \quad v_0 = 0; \quad \partial p_0/\partial x = -8. \quad (48)$$

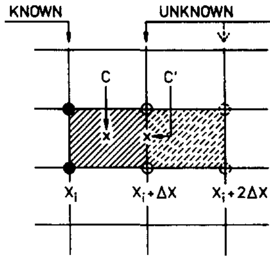

FIG. 2. Details of the net rectangle for the Box scheme difference equations.

Table 3. Analytical and numerical results for $u_{1\eta}$ at $x/Pr = 3$, thermal boundary condition: q_w = const.

r	Analytical equation (49)	Numerical	Percentage deviation (%)
0.0	0.5455	0.5449	0.11
0.2	0.4544	0.4538	0.13
0.4	0.2216	0.2212	0.18
0.6	−0.0397	−0.0399	0.50
0.8	−0.1689	−0.1690	0.06
1.0	0.0	0.0	—

All other equations, the zero-order energy equation (21) and all first-order equations, equations (22)–(31), were solved by the so-called 'Box scheme' method [6]. This method is implicit with respect to variations normal to the tube axis and second-order accurate in all variables on arbitrary non-uniform nets. Though it was originally applied to boundary layers it is easily transferred to the tube flow problem.

There have been no stability problems (actually the scheme is claimed to be unconditionally stable) but in certain situations slight oscillations around the smooth solution occurred. They could be suppressed completely by a very simple 'trick' first proposed by Wickern [7] and afterwards successfully applied to a great number of different flow situations. It is shortly explained in Fig. 2. In the conventional Box scheme the differential equations are approximated by centring about the midpoint of each box, point C in Fig. 2. As a consequence you easily get oscillations in the x-directions as may be illustrated by the v-velocity which should be zero for all x in fully developed flow. If, by mistake, v is nonzero at x_i, say v^+, the differential equation is nevertheless fulfilled by its difference approximation, if $v = -v^+$ at $x_i+\Delta x$, since the central difference $v = \frac{1}{2}(v^+ + (-v^+)) = 0$! This problem is circumvented by shifting the central point from C to C′ (see Fig. 2). Now $v = 0$ is enforced at $x_i+\Delta x$ and no oscillations occur. Nothing has to be changed in the formal system if one proceeds with the following two steps. (1) Choose a box of twice the x-step size, $2\Delta x$ in Fig. 2. This gives results at $x_i+2\Delta x$. (2) Take the mean average of all quantities at x_i and $x_i+2\Delta x$, which provides the unknown quantities at $x_i+\Delta x$.

In addition to this procedure the first step out of $x = 0$ was accomplished by an analytical Leveque-like step, see for example Worsoe-Schmidt [8].

As a test case which could also be used to find the proper grid sizes we could refer to the analytical results for the q_w = const. case in ref. [1]. The downstream limit ($x \to \infty$) for $u_{1\eta}$, cf. equation (14), for example reads [1]

$$u_{1\eta} = \tfrac{2}{11}(-2r^6 + 12r^4 - 13r^2 + 3); \quad q_w = \text{const.} \tag{49}$$

Various test calculations showed that grid size independence (variations of typical, selected quantities smaller than 10^{-4}) is given for

$$\Delta r \leqslant 0.02; \quad \Delta x \leqslant 0.01 Pr_w. \tag{50}$$

In Table 3 the numerical results at $x/Pr = 3$ are compared with those according to equation (49). Deviations are less than 0.5% which we decided to tolerate since the analytical results hold for $x \to \infty$ whereas the numerical results were found at $x/Pr = 3$ after 300 steps in the x-direction.

Applying this numerical method to the T_w = const. case under consideration provides us with the functions $\bar{T}_0$, $\bar{T}_{1\alpha}$, $u_{1\alpha}$, $v_{1\alpha}$ all x and r dependent with the Prandtl number as a parameter. To avoid extensive data documentation we only present the data for $x \to \infty$ (fully developed flow) incorporated in the exponents of the property ratio method—as far as it is possible to do that.

In the following we show how this can be accomplished.

First Newton's law $\tau_w^* = \eta_w^*(\partial u^*/\partial y^*)_w$ and Fourier's law $q_w^* = -\lambda_w^*(\partial T^*/\partial y^*)_w$ are written in dimensionless form bearing in mind that $\partial/\partial y^* = -\partial/\partial r^*$, see Fig. 1 for the coordinates y^* and r^*. The skin friction and heat transfer relations in terms of f and Nu, respectively, are

$$f \equiv \frac{2\tau_w^*}{\rho_w^* U_w^{*2}} = -2\left.\frac{\partial u}{\partial r}\right|_w Re_w^{-1} \tag{51}$$

$$Nu \equiv \frac{2q_w^* R^*}{\lambda_w^*(T_w^* - T_B^*)} = -2\left.\frac{\partial \bar{T}}{\partial r}\right|_w \bar{T}_B^{-1}. \tag{52}$$

Next

$$u = u_0 + \varepsilon(K_\eta u_{1\eta} + K_\rho u_{1\rho}) + O(\varepsilon^2), \; \bar{T} = \bar{T}_0 + \ldots$$

as well as an appropriate expansion for $\bar{T}_B$ will be inserted into equations (51) and (52). With $\bar{T}_B$ according to equation (17)

$$\bar{T}_B = \bar{T}_{B0} + \varepsilon(K_\eta \bar{T}_{B1\eta} + K_\rho \bar{T}_{B1\rho} + K_\lambda \bar{T}_{B1\lambda} + K_c \bar{T}_{B1c}) + O(\varepsilon^2) \tag{53}$$

the asymptotic results for f and Nu follow immediately. They can be written in a very clearly arranged form if they are referred to their constant property limits f_{cp} and Nu_{cp} (cf. equations (51) and (52))

$$f_{cp} Re_w = -2 \left.\frac{\partial u_0}{\partial r}\right|_w = 8 \tag{54}$$

$$Nu_{cp} = -2 \left.\frac{\partial \bar{T}_0}{\partial r}\right|_w \bar{T}_{B0}^{-1} = Nu_{cp}(Pr_w). \tag{55}$$

The final results are

$$\frac{f\, Re_w}{(f\, Re_w)_{cp}} = 1 - \varepsilon(K_\eta A_\eta + K_\rho A_\rho) + O(\varepsilon^2);$$

$$A_\alpha = -\frac{1}{4}\left.\frac{\partial u_{1\alpha}}{\partial r}\right|_w; \quad \alpha = \eta, \rho \tag{56}$$

$$\frac{Nu}{Nu_{cp}} = 1 + \varepsilon(K_\eta B_\eta + K_\rho B_\rho + K_\lambda B_\lambda + K_c B_c) + O(\varepsilon^2) \tag{57}$$

$$B_\alpha = \frac{\left.\dfrac{\partial \bar{T}_{1\alpha}}{\partial r}\right|_w}{\left.\dfrac{\partial \bar{T}_0}{\partial r}\right|_w} - \frac{\bar{T}_{B1\alpha}}{\bar{T}_{B0}}; \quad \alpha = \eta, \rho, \lambda, c.$$

The bulk temperatures $\bar{T}_{B0}$ and $\bar{T}_{B1\alpha}$ in equations (56) and (57) immediately follow from the definition of $\bar{h}_B$

$$\bar{h}_B = \int_0^1 \rho u \bar{h} r \, dr \tag{58}$$

together with the enthalpy/temperature relations (32) and (33). Inserting $\rho = \rho_0 + \ldots$, $u = u_0 + \ldots$ and $\bar{h} = \bar{h}_0 + \ldots$ into equation (58) we obtain

$$\bar{T}_{B0} = 2 \int_0^1 u_0 \bar{T}_0 r \, dr \tag{59}$$

$$\bar{T}_{B1\eta} = 2 \int_0^1 (\bar{T}_0 u_{1\eta} + u_0 \bar{T}_{1\eta}) r \, dr \tag{60}$$

$$\bar{T}_{B1\rho} = 2 \int_0^1 [\bar{T}_0(u_0 \bar{T}_0 + u_{1\rho}) + u_0 \bar{T}_{1\rho}] r \, dr \tag{61}$$

$$\bar{T}_{B1\lambda} = 2 \int_0^1 u_0 \bar{T}_{1\lambda} r \, dr \tag{62}$$

$$\bar{T}_{B1c} = -\tfrac{1}{2}\bar{T}_{B0}^2 + 2 \int_0^1 (u_0 \bar{T}_{1c} + \tfrac{1}{2} u_0 \bar{T}_0^2) r \, dr. \tag{63}$$

From the final results, equations (56) and (57), we find the exponents of the property ratio method by comparing these equations with the property ratio formulae

$$\frac{f\, Re_w}{(f\, Re_w)_{cp}} = \left(\frac{\eta_w^*}{\eta_B^*}\right)^{n_\eta} \left(\frac{\rho_w^*}{\rho_B^*}\right)^{n_\rho} \tag{64}$$

$$\frac{Nu}{Nu_{cp}} = \left(\frac{\eta_w^*}{\eta_B^*}\right)^{m_\eta} \left(\frac{\rho_w^*}{\rho_B^*}\right)^{m_\rho} \left(\frac{\lambda_w^*}{\lambda_B^*}\right)^{m_\lambda} \left(\frac{c_{pw}^*}{c_{pB}^*}\right)^{m_c}. \tag{65}$$

Incorporating the following expansion for a general property α^*:

$$\left[\frac{\alpha_w^*}{\alpha_B^*}\right]^\beta \equiv \alpha_B^{-\beta} = [1 + \varepsilon K_\alpha \bar{T}_{B0} + O(\varepsilon^2)]^{-\beta} = 1 - \varepsilon K_\alpha \bar{T}_{B0} \beta + O(\varepsilon^2) \tag{66}$$

the property ratio formulae in their asymptotic versions are

$$\frac{f\, Re_w}{(f\, Re_w)_{cp}} = 1 - \varepsilon(n_\eta K_\eta \bar{T}_{B0} + n_\rho K_\rho \bar{T}_{B0}) + O(\varepsilon^2) \tag{67}$$

$$\frac{Nu}{Nu_{cp}} = 1 - \varepsilon(m_\eta K_\eta \bar{T}_{B0} + m_\rho K_\rho \bar{T}_{B0} + m_\lambda K_\lambda \bar{T}_{B0} + m_c K_c \bar{T}_{B0}) + O(\varepsilon^2). \tag{68}$$

Comparing equations (64), (67) and (65), (68), respectively, provides us with the exponents of the property ratio method (for a detailed discussion of higher order effects see ref. [1])

$$n_\alpha = \frac{1}{4\bar{T}_{B0}} \left.\frac{\partial u_{1\alpha}}{\partial r}\right|_w \quad \text{with } \alpha = \eta, \rho \tag{69}$$

$$m_\alpha = \frac{\bar{T}_{B1\alpha}}{\bar{T}_{B0}^2} - \frac{\left.\dfrac{\partial \bar{T}_{1\alpha}}{\partial r}\right|_w}{\left.\dfrac{\partial \bar{T}_0}{\partial r}\right|_w \bar{T}_{B0}} \quad \text{with } \alpha = \eta, \rho, \lambda, c. \tag{70}$$

As mentioned before not all exponents n_α, m_α turned out to be constants for a fixed Prandtl number in the limit $x \to \infty$. For Prandtl numbers above about $Pr_w = 0.5$ all are constants. Below a Prandtl number of about 0.5 the exponents n_η, m_η and n_ρ, m_ρ are x-dependent, the stronger the lower the Prandtl number is.

In Table 4 the exponents are listed for Prandtl numbers greater than $Pr_w = 0.5$. Comparing them with the q_w = const. case in ref. [1] shows that they all have the same sign in both cases and most of them differ by less than about 20%. (For comparing n_η, n_ρ and m_λ note that the definitions of the exponents must be rearranged according to the definitions of $\tilde{f}$ and $\tilde{Nu}$ in ref. [1].)

For Prandtl numbers below about $Pr = 0.5$ the numerical results showed a decreasing tendency for constant values of the exponents n_η, m_η, n_ρ and m_ρ in the downstream limit $x \to \infty$ (which for all other exponents was reached numerically at about $x/Pr_w = 3$). In Fig. 3 the exponents n_η and m_η are shown for decreasing Prandtl numbers, those for the density (n_ρ, m_ρ) exhibit the same trends.

As a consequence we must conclude that the property ratio method—which assumes x-independent exponents—fails for Prandtl numbers below $Pr_w \approx 0.5$ in the T_w = const. case. From an asymptotic point of view things are even more restrictive, since the property ratio method only holds in the limit of $Pr_w \to \infty$ asymptotically as we will try to verify in the following section.

Table 4. Exponents in the property ratio method for $T_w^* = \text{const.}$, equations (69) and (70)

Pr_w	n_η	n_ρ	m_η	m_ρ	m_λ	m_c
$\to\infty$	−0.4191	1.0000	−0.1225	0.3286		
100	−0.4197	0.9972	−0.1228	0.3277		
10	−0.4262	0.9720	−0.1253	0.3196		
5	−0.4329	0.9425	−0.1282	0.3102	↑	↑
2	−0.4554	0.8467	−0.1376	0.2788	−0.7743	0.2743
1	−0.5014	0.6597	−0.1569	0.2149	↓	↓
0.7	−0.5524	0.4610	−0.1786	0.1435		
0.5	−0.6475	0.1073	−0.2196	0.0097		

For low Prandtl numbers the influence of variable properties cannot be represented by constant exponents, so that the property ratio method is not adequate. Instead one should go back to the final results (56) and (57), respectively, which for convenience will be rewritten as

$$\frac{f\,Re_w}{(f\,Re_w)_{cp}} = 1+\varepsilon\sum_\alpha K_\alpha A_\alpha(x, Pr_w)+O(\varepsilon^2);$$

$$A_\alpha \text{ according to equation (56)} \qquad (71)$$

$$\frac{Nu}{Nu_{cp}} = 1+\varepsilon\sum_\alpha K_\alpha B_\alpha(x, Pr_w)+O(\varepsilon^2);$$

$$B_\alpha \text{ according to equation (57).} \qquad (72)$$

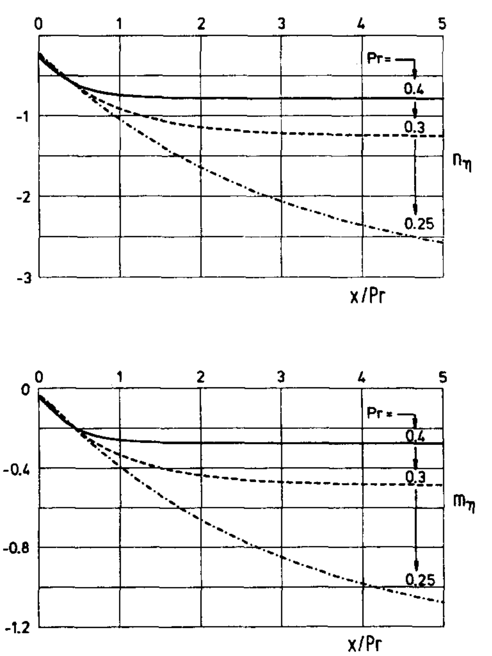

FIG. 3. Property ratio exponents n_η, m_η for Prandtl numbers $Pr_w \leqslant 0.5$.

In Fig. 4 the auxiliary functions A_α and B_α are given for the small Prandtl numbers, $Pr = 0.1, 0.05, 0.01$. Those for B_λ and B_c collapse for all Prandtl numbers. (A careful study of this phenomena leads to the fact that m_λ and m_c are constants for all Prandtl numbers and $x \to \infty$ in contrast to what holds for η and ρ. The reason is that there are no first-order momentum equations for λ and c_p, see also Table 4.)

5.2. *Separation of variables approach*

At the beginning of Section 5 we mentioned our first assumption that the zero-order x-dependence of the temperature $\bar{T}_0$ is carried to the higher order equations. This assumption is supported by equation (69) for example. The exponents n_η and n_ρ can be independent of x only if $u_{1\eta}$, $u_{1\rho}$ exhibit the same x-dependence as $\bar{T}_{B0}$, i.e. $\exp[-\Lambda_1^2 x/Pr_w]$.

As an example we will illustrate the determination of n_η based on the assumption of a unique x-dependence of all zero- and first-order quantities. According to this approach we assume ($\Lambda_1^2 = 3.6568$)

$$u_{1\eta} = \bar{u}_{1\eta}(r)\exp[-\Lambda_1^2 x/Pr_w] \qquad (73)$$

$$v_{1\eta} = v_{1\eta}(r)\exp[-\Lambda_1^2 x/Pr_w] \qquad (74)$$

$$\frac{\partial p_{1\eta}}{\partial x} = \bar{C}_\eta \exp[-\Lambda_1^2 x/Pr_w]. \qquad (75)$$

Inserting equations (73)–(75) together with equation (46) for $\bar{T}_0$ into the first-order momentum equation, equation (23), results in an ordinary differential equation for $\bar{u}_{1\eta}(r)$

$$\left(-\frac{2\Lambda_1^2}{Pr_w}\right)\left[(1-r^2)\bar{u}_{1\eta}+2\int_0^r \bar{u}_{1\eta} r\,dr\right] = \bar{C}_\eta + [r\bar{u}'_{1\eta} - 4C_{01}r^2\bar{T}_{0r}]'/r. \qquad (76)$$

Here a prime denotes a derivative with respect to r, the integral on the left-hand side comes in through continuity equation (22). The boundary conditions are

$$\bar{u}_{1\eta} = 0 \quad \text{at } r = 1 \qquad (77)$$

$$\partial\bar{u}_{1\eta}/\partial r = 0 \quad \text{at } r = 0. \qquad (78)$$

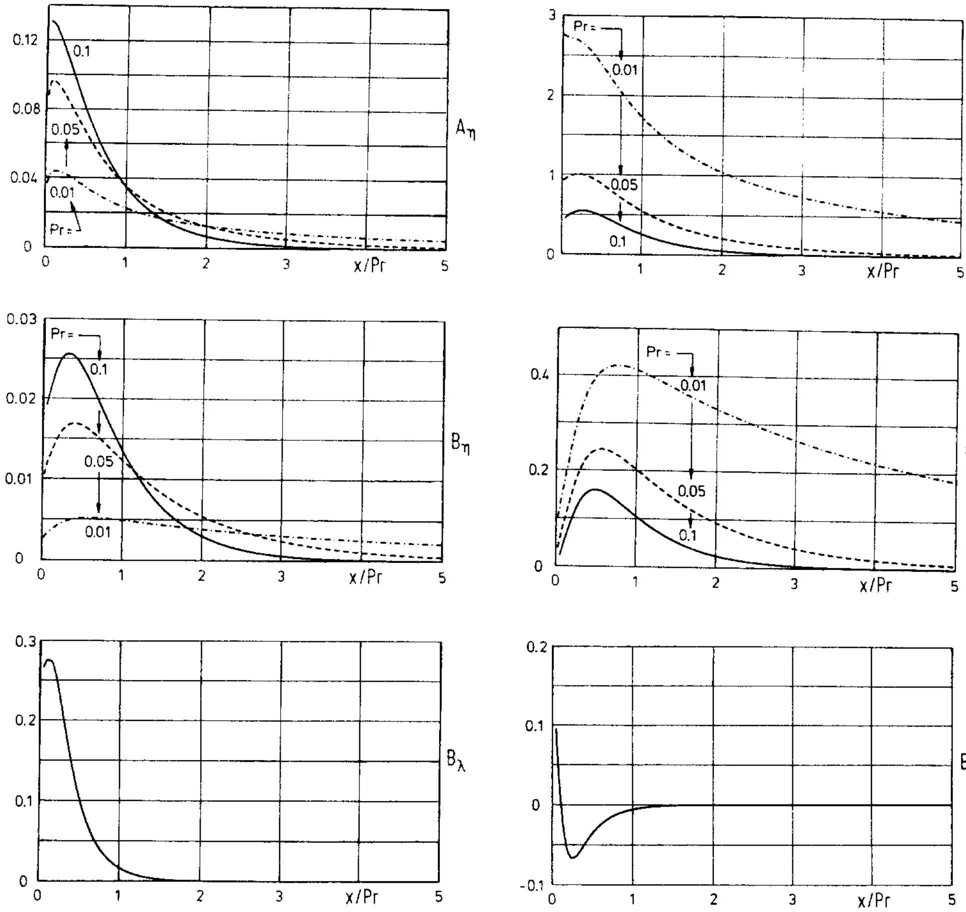

FIG. 4. Auxiliary functions A_η, A_p and B_η, B_p, B_λ, B_c; see equations (71) and (72).

The constant $\bar{C}_\eta$ is determined through the integral condition of the constant mass flux which in its asymptotic form imposes the condition

$$\int_0^1 \bar{u}_{1\eta} r \, dr = 0 \tag{79}$$

on the velocity $\bar{u}_{1\eta}$.

Solving equation (76) by a standard Runge–Kutta technique for ordinary differential equations is straightforward as soon as a specific Prandtl number is fixed. With $\bar{u}_{1\eta}(r)$ we can immediately calculate n_η according to equation (69). In Fig. 5 the results are compared with those from the completely numerical approach of the previous section. For moderate and large Prandtl numbers ($Pr \geqslant 0.5$) the agreement is very good.

For $Pr_w = 10$, for example, n_η differs by less than 0.2% ($n_\eta = -0.4262$ from the numerical approach and $n_\eta = -0.4254$ based on the solution of equation (76)).

For small Prandtl numbers assumptions (73)–(75) fail completely which result in a somewhat curious solution of equation (76) with a sequence of singular points for decreasing Prandtl numbers. From our numerical results in Section 5.1 we known that for low Prandtl numbers $u_{1\eta}$ does not behave according to assumptions (73)–(75). But what is the physical reason for that?

The first-order momentum equation (76) explicitly shows the main Prandtl number influence. It comes in through the inertia forces on the left-hand side of the momentum equation. For moderate to high

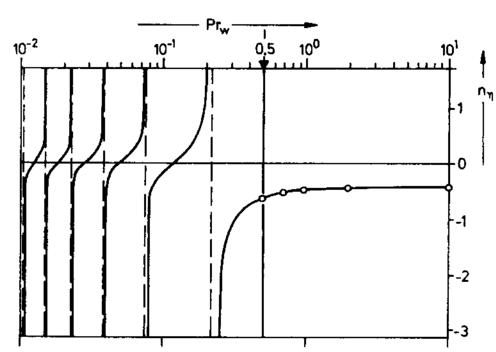

FIG. 5. Property ratio exponent n_η: ——, separation approach; ○, numerical approach.

Prandtl numbers the inertia forces are small compared with the pressure and viscous forces on the right-hand side. For an infinite Prandtl number they vanish completely since they are of the order of $O(Pr_w^{-1})$. Obviously the first-order solutions are of the assumed type (73)–(75) as long as the balance of forces is that between pressure and viscous forces, with negligible inertia forces. The solution for u_{1_η} is no longer the product of an x- and r-dependent part when inertia forces have to be accounted for, obviously because inertia forces cannot be constant over the cross-section of a pipe. They are zero at the wall and non-zero away from the wall.

From these considerations we conclude that strictly speaking constant property exponents only exist in the limit $Pr_w^{-1} = 0$, i.e. for an infinite Prandtl number. For Prandtl numbers above 0.5 the influence of the inertia forces is small enough that the exponents are not effected within the accuracy limits of Table 4 or Fig. 5. So for practical applications they can be used as 'nearly constant'.

6. CONCLUSIONS

In this study we extended a method to account for variable property effects to the thermal boundary condition $T_w^* = \text{const.}$ In an earlier study laminar tube flow was investigated for the boundary condition $q_w^* = \text{const.}$ In that study a well-known empirical method, the property ratio method, was established as an analytical method [1].

Now it turned out that this method can be applied in the $T_w^* = \text{const.}$ case only under certain conditions. For practical applications it may be a useful method with Prandtl numbers above about 0.5. For smaller Prandtl numbers it is no longer adequate and for Prandtl numbers as small as those for liquid metals ($Pr \approx 10^{-3}$) it fails completely.

This conclusion could only be drawn on the basis of an analytical analysis of the empirical method.

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EFFET DES PROPRIETES VARIABLES SUR LE TRANSFERT DE QUANTITE DE MOUVEMENT ET DE CHALEUR DANS UN TUBE A TEMPERATURE PARIETALE CONSTANTE

Résumé—On étudie l'effet des propriétés variables sur le transfert de quantité de mouvement et de chaleur en appliquant une méthode asymptotique déjà utilisée avec la condition limite thermique $q_w = \text{const.}$ Bien que les changements sont seulement dans les conditions aux limites, les deux cas sont très différents sous divers aspects. La méthode empirique utilisée dans les deux cas (méthodes du rapport de propriété) n'est pas uniquement valable dans le cas $T_w = \text{const.}$ de cette étude.

DER EINFLUSS VARIABLER STOFFEIGENSCHAFTEN AUF IMPULS- UND WÄRMEÜBERTRAGUNG IN EINEM ROHR MIT KONSTANTER WANDTEMPERATUR

Zusammenfassung—Mit einer asymptotischen Methode, die bisher bei konstanter Wärmestromdichte verwendet worden ist, wird nun der Einfluß variabler Stoffeigenschaften auf Impuls- und Wärmeübertragung bei konstanter Wandtemperatur untersucht. Obwohl die Änderungen nur in der Wahl der thermischen Randbedingungen bestehen, ergibt sich, daß beide Fälle in unterschiedlicher Hinsicht sehr verschieden sind. Als ein Hauptergebnis erweist sich, daß eine in beiden Fällen empirisch genutzte Methode (Methode der Stoffwert-Verhältnisse) in dem hier betrachteten Fall $T_w = \text{const.}$ eindeutig nicht gültig ist.

ВЛИЯНИЕ ПЕРЕМЕННЫХ ХАРАКТЕРИСТИК НА ПЕРЕНОС ИМПУЛЬСА И ТЕПЛА В ТРУБЕ С ПОСТОЯННОЙ ТЕМПЕРАТУРОЙ СТЕНОК

Аннотация—С помощью асимптотического метода, использовавшегося ранее при тепловом граничном условии $q_w = \text{const.}$, исследуется влияние переменных характеристик на перенос импульса и тепла. Несмотря на то, что отличие имеет место только в тепловых граничных условиях, оба случая во многих аспектах совершенно различны. Основным результатом исследования является установление факта, что эмпирический метод, используемый в обоих случаях при $T_w = \text{const.}$ не является однозначным.